**Exercise 1**  
Consider the (free-field) Klein-Gordon equation given by
\[ \Box \phi(x, t) - \frac{m^2 c^2}{\hbar^2} \phi(x, t) = 0, \quad \Box = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \]  
and let \( \phi(x, t) \) be a solution to it. Defining \( \psi(x, t) \) such that
\[ \phi(x, t) = \psi(x, t) e^{-imc^2 t/\hbar}, \]  
a) determine under which condition \( \psi(x, t) \) will satisfy the non-relativistic Schrödinger equation
\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(x, t). \]  
Give a physical interpretation when \( \phi \) is a plane wave solution.  

**Exercise 2**  
**Charged harmonic oscillator in a variable electric field**  
A one-dimensional harmonic oscillator is composed of a particle of mass \( m \), charge \( q \) and potential energy \( V(\hat{x}) = \frac{1}{2} m \omega^2 \hat{x}^2 \). We assume that the particle is placed in an electric field \( E(t) \) parallel to the \( x \)-axis and time-dependent, so that the potential energy:
\[ \hat{W}(t) = -qE(t) \hat{x} \]  
has to be added to \( V(\hat{x}) \). Let \( |\psi(0)\rangle \) be the state of the system at \( t = 0 \).

a) Write the hamiltonian \( \hat{H}(t) \) of the particle in terms of the operators \( \hat{a} \) and \( \hat{a}^\dagger \) (annihilation and creation operators of the simple harmonic oscillator). Evaluate the commutators of \( \hat{a} \) and \( \hat{a}^\dagger \) with \( \hat{H}(t) \).

b) Let \( \alpha(t) \) be the number defined by:
\[ \alpha(t) = \langle \psi(t) | \hat{a} | \psi(t) \rangle \]  
where \( |\psi(t)\rangle \) is the normalized state vector at time \( t \) of the particle under study. Using the previous results, show that \( \alpha(t) \) satisfies the differential equation:
\[ \frac{\partial}{\partial t} \alpha(t) = -i\omega \alpha(t) + i\lambda(t) \]  
where
\[ \lambda(t) = \frac{q}{\sqrt{2m\hbar\omega}} E(t). \]  
Integrate this differential equation. At time \( t \), what are the mean values of the position and momentum of the particle?  

(1 Point)
c) The ket $|\phi(t)\rangle$ is defined by:

$$|\phi(t)\rangle = (\hat{a} - \alpha(t))|\psi(t)\rangle$$  \hspace{1cm} (7)

where $\alpha(t)$ is the value calculated in b. Using the results of questions a and b, show that the evolution of $|\phi(t)\rangle$ is given by:

$$i\hbar \frac{\partial}{\partial t}|\phi(t)\rangle = (\hat{H}(t) + \hbar \omega)|\phi(t)\rangle$$ \hspace{1cm} (8)

How does the form of $|\phi(t)\rangle$ vary with time?  \hspace{1cm} (1 Point)

d) Assuming that $|\psi(0)\rangle$ is an eigenvector of $\hat{a}$ with eigenvalue $\alpha(0)$, show that $|\psi(t)\rangle$ is also an eigenvector of $\hat{a}$, and evaluate its eigenvalue. Find at time $t$ the mean value of the unperturbed Hamiltonian

$$\hat{H}_0 = \hat{H}(t) - \hat{W}(t)$$ \hspace{1cm} (9)

as a function of $\alpha(0)$. Give the root-mean-square deviations

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle_t - \langle \hat{x} \rangle_t^2}, \hspace{1cm} \Delta p = \sqrt{\langle \hat{p}^2 \rangle_t - \langle \hat{p} \rangle_t^2}, \hspace{1cm} \Delta H_0 = \sqrt{\langle \hat{H}_0^2 \rangle_t - \langle \hat{H}_0 \rangle_t^2};$$ \hspace{1cm} (10)

how do they change in time? \hspace{1cm} (2 Points)