Exercise 12  Photons with circular polarization
Consider the two modes at wave vector \( k \) and linear polarizations \( \epsilon_1 \) and \( \epsilon_2 \), such that \( k \cdot \epsilon_1 = k \cdot \epsilon_2 = \epsilon_1 \cdot \epsilon_2 = 1 \).
The energy is \( \hat{H} = \hbar \omega (\hat{a}_1^\dagger a_1 + \hat{a}_2^\dagger a_2) \) where \( \omega = c|k| \) and \( a_j \) is the annihilation operator of a photon with wave-vector \( k \) and polarization \( \epsilon_j \), satisfying the commutation relation
\[
[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}.
\] (1)

a) Take the modes with circular polarization
\[
\hat{a}_+ = -\frac{\hat{a}_1 + i\hat{a}_2}{\sqrt{2}}, \\
\hat{a}_- = \frac{\hat{a}_1 - i\hat{a}_2}{\sqrt{2}}
\] (2)
show that
\[
[\hat{a}_\pm, \hat{a}_\pm^\dagger] = 1, \\
[\hat{a}_\mp, \hat{a}_\mp^\dagger] = 0,
\] (3)
and the Hamiltonian can be rewritten as \( \hat{H} = \hbar \omega (\hat{a}_+^\dagger a_+ + \hat{a}_-^\dagger a_-) \). (1 Point)
b) Perform an infinitesimal rotation about \( k \) of \( \delta \phi \ll 1 \). Write \( \epsilon_1', \epsilon_2' \) and show that \( \delta \epsilon'_\pm = \mp i\delta \phi \epsilon_\pm \), where \( \delta \epsilon'_\pm = \epsilon'_\pm - \epsilon_\pm \) therefore the photon with \( \epsilon_\pm \) has spin component \( \pm \hbar \). (1 Point)
c) Discuss why the chemical potential of the photons is zero. (1 Point)

Exercise 13  Properties of the gamma matrices
The Dirac-Hamiltonian operator is given by
\[
\hat{H} = c \alpha \cdot p + \alpha_t mc^2
\] (4)
with \( \alpha_\mu = (\alpha_x, \alpha_y, \alpha_z, \alpha_t) \). It holds that \( \alpha_\mu^\dagger = \alpha_\mu \), \( \text{Tr} \{ \alpha_\mu \} = 0 \) as well as \( \{ \alpha_\mu, \alpha_\nu \} = 2 \delta_{\mu \nu} \), with the anticommutator defined as \( \{ A, B \} = AB + BA \). Beside, the Dirac Equation in the van-der-Waerden-Form is given by
\[
\left( \gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar} \right) \psi(\vec{x}, t) = 0,
\] (5)
where the \( \gamma \)-matrices are defined as \( \gamma_j = -i \alpha_t \alpha_j \) for \( j = 1, 2, 3 \) and \( \gamma_4 = \alpha_t \). Show the following properties, without explicitly writing the gamma matrices:
1. the anticommutators \( \{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu\nu} \) hold,
2. the \( \gamma_\mu \) are hermitian,
3. the trace of the \( \gamma_\mu \) vanishes.