Exercise 16  
**Interpretation of the Dirac matrix**

Consider the operator $\hat{x}$, position operator of an electron with mass $m$ and charge $e$, such that $[\hat{x}_j, \hat{p}_k] = i\hbar \delta_{jk}$ with $j, k = 1, 2, 3$.

a) Given $\hat{H} = c \hat{\alpha} \cdot \hat{p} + \hat{\alpha} tc^2$, with $\{ \hat{\alpha}_\mu, \hat{\alpha}_\nu \} = 2 \delta_{\mu\nu}, \mu, \nu = 1, 2, 3, 4$ and $\hat{\alpha}_\mu = \hat{\alpha}_\mu^\dagger$, show that (in the Heisenberg picture)

$$\frac{d\hat{x}_k}{dt} = c \hat{\alpha}_k,$$

therefore the Dirac matrices $\hat{\alpha}_k$ are velocity components.  

$(0.5 \text{ Points})$

b) The magnetic moment of the electrons is

$$\hat{\mu} = \frac{e}{2} \hat{r} \times \hat{\alpha}.$$  

Show that for an eigenstate of the energy at eigenvalue $E$, the expectation value of $\hat{\mu}$ is

$$\langle \hat{\mu} \rangle = \frac{ec}{2E}(L + 2S)$$  

where $L = \langle \hat{r} \times \hat{p} \rangle$ is the orbital momentum and $S = \hbar \langle \hat{\sigma} \rangle / 2$ is the spin operator.

In order to do this, consider an operator $\langle O \rangle$ and calculate $\langle \hat{O} \rangle = \langle E | \hat{O} | E \rangle$, with $|E\rangle$ a normalized eigenstate of the energy such that $\langle E | E \rangle = 1$, $\hat{H} | E \rangle = E | E \rangle$. Show that it can be written as:

$$\langle \hat{O} \rangle = \frac{1}{2E} \{ \hat{H}, \hat{O} \},$$

with $E \neq 0$ and $\{, \}$ being the anticommutator. Use Eq. (4) to derive Eq. (3).  

$(2 \text{ Points})$

c) Show that

$$\frac{d\hat{\alpha}}{dt} = \frac{2}{i\hbar} (\hat{\alpha} \hat{H} - c \hat{p}),$$

$$\frac{d\hat{p}}{dt} = 0,$$

$$\frac{d\hat{H}}{dt} = 0.$$  

$(0.5 \text{ Points})$

Exercise 17  
**Zitterbewegung**

We will calculate $\langle \hat{x}(t) \rangle$ of a free electron and compare this dynamics with the one of the non-relativistic motion.
In order to do this we want to determine
\[
\langle \hat{x}(t) \rangle = \int_V d\mathbf{x}' |\psi(\mathbf{x}', t)|^2
\]  
(6)
with \(|\psi(\mathbf{x}', t)|^2 = \psi^\dagger(\mathbf{x}', t)\psi(\mathbf{x}', t)\). Here
\[
\psi(\mathbf{x}, t) = \sum_{j,p} c_{j,p} \psi^{(j)}(\mathbf{x}, t)
\]  
(7)
\[
\psi^{(j)}(\mathbf{x}, t) = \frac{1}{\sqrt{V}} u^{(j)}(p) e^{i(p \cdot \mathbf{x} - E^{(j)} t)/\hbar}, \quad j = 1, 2, 3, 4.
\]  

The \(u^{(j)}(p)\) are given by:

\[
u^{(1)}(p) = N \begin{pmatrix} 1 \\ 0 \\ \frac{c p_z}{E + mc^2} \\ \frac{c (p_x + ip_y)}{E + mc^2} \end{pmatrix}, \quad u^{(2)}(p) = N \begin{pmatrix} 0 \\ 1 \\ \frac{c (p_x - ip_y)}{E + mc^2} \\ \frac{c p_z}{E + mc^2} \end{pmatrix},
\]

\[
u^{(3)}(p) = N \begin{pmatrix} 0 \\ 1 \\ \frac{-c p_z}{|E| + mc^2} \\ \frac{-c (p_x + ip_y)}{|E| + mc^2} \end{pmatrix}, \quad u^{(4)}(p) = N \begin{pmatrix} 0 \\ 1 \\ \frac{-c (p_x - ip_y)}{|E| + mc^2} \\ \frac{-c p_z}{|E| + mc^2} \end{pmatrix}
\]
with normalization \(N = \sqrt{(|E| + mc^2)/(2|E|)}\).

Here \(p_x, p_y, p_z\) are the components of the momentum \(p\), \(m\) is the mass, \(V\) is the volume and \(E^{(j)} = \pm \sqrt{p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m^2 c^4}\) is the energy (such that \(E^{(j)} > 0\) for \(j = 1, 2\) and \(E^{(j)} < 0\) for \(j = 3, 4\)).

a) Use Heisenberg picture and Eqs. (5) to show that
\[
\ddot{\hat{\alpha}}_k(t) = \frac{2}{i\hbar} \left( \hat{\alpha}_k \hat{H} \right).
\]  
(8)
Demonstrate that the solution of this equation of motion reads
\[
\hat{\alpha}_k(t) = \hat{B}_0 + \hat{\alpha}_k(0) e^{-2i\hat{H} t/\hbar} \left( \frac{i\hbar}{2} \right) \hat{H}^{-1}, \quad \hat{B}_0 = c \hat{p}_k \hat{H}^{-1}.
\]  
(2 Points)

b) Determine \(\hat{x}_k(t)\) using Eqs. (1) and (5) and show that
\[
\hat{x}_k(t) = \hat{x}_k(0) + (c^2 \hat{p}_k \hat{H}^{-1}) t + \frac{i\hbar c}{2} \left( \hat{\alpha}_k(0) - c \hat{p}_k \hat{H}^{-1} \right) \hat{H}^{-1} \left( e^{-2i\hat{H} t/\hbar} - 1 \right).
\]  
(1 Point)
c) Using Eq. (7) calculate explicitly

\[
\langle c^2 \hat{p}_k \hat{H}^{-1} \rangle = \int_V d\mathbf{x}' \psi^{\dagger}(\mathbf{x}') c^2 \hat{p}_k \hat{H}^{-1} \psi(\mathbf{x}')
\]

\[
\langle \frac{i\hbar c}{2} (\hat{\alpha}_k(0) - c\hat{p}_k \hat{H}^{-1}) \hat{H}^{-1} \left( e^{-2i\hat{H}t/\hbar} - 1 \right) \rangle = \int_V d\mathbf{x}' \psi^{\dagger}(\mathbf{x}') \frac{i\hbar c}{2} (\hat{\alpha}_k(0) - c\hat{p}_k \hat{H}^{-1}) \hat{H}^{-1} \left( e^{-2i\hat{H}t/\hbar} - 1 \right) \psi(\mathbf{x}').
\]  

(11)

What happens if \(c_{j,p} = 0 \forall p\) and \(j = 3, 4\)? (2 Point)

Hint: use the fact that

\[
u^{(j)}(\mathbf{p})^\dagger \hat{\alpha}_k(0) u^{(j')}(\mathbf{p}) = \begin{cases} 
\frac{p_k c}{E} \delta_{jj'}, & \text{for } j, j' = 1, 2 \\
-\frac{p_k c}{E} \delta_{jj'}, & \text{for } j, j' = 3, 4 \\
\frac{|E|}{\alpha_{jj'}} & \text{for } j = 1, 2 \text{ and } j' = 3, 4 \text{ or } j' = 1, 2 \text{ and } j = 3, 4.
\end{cases}
\]

Calculate \(a_{jj'}^{(k)}\) for \(k = 1\) and verify it is different from 0.