Question 1  An elastically bound electron

a) Newton’s equation of motion for a charged particle of mass $m$ and charge $e$ which is elastically bound to the origin is given by

$$m \ddot{r} + fr = 0$$

(1)

where $f$ is the spring constant and the oscillator eigenfrequency is $\omega_0^2 = \frac{f}{m}$.

Write the energy $W$ of the system and the general solution.

(1 Point)

b) The energy which is dissipated by the electron is given by

$$S = -\frac{dW}{dt} = \frac{2e^2 \dot{r}^2}{3c^3 \overline{\dot{r}^2}} ,$$

(2)

where $c$ is the speed of light, $\dot{r}$ is the acceleration and $\overline{\dot{r}^2}$ is its time average $\overline{\dot{r}^2} = \frac{1}{\tau} \int_0^\tau \dot{r}(t) dt$, where $\tau$ is the period. Show that for quasiperiodic motion

$$\frac{dW}{dt} = -\gamma W ,$$

(3)

and determine the damping coefficient $\gamma$.

(1 Point)

c) Assuming quasi-periodic motion, we now include the emitted radiation from the accelerating electron such that the new equation of motion becomes

$$m \ddot{r} + fr = \mathcal{R}$$

(4)

where $\mathcal{R}$ is the force which gives rise to a change in the total energy.

Using Eq.(2) find $\mathcal{R}$ in terms of $\dot{r}$ by first multiplying Eq.(4) by $\dot{r}$. Then using $r = U e^{i\omega_0 t}$, where $U$ is a complex vector, find a compact expression for the frequency of these oscillations. Where the time after which the energy of the emitting atom is $1/e$ its initial value is given by

$$T = \frac{1}{\gamma} = 4 \times 10^{-7} \text{s}$$

(5)

and $\omega_0 = 2\pi \times 10^{14} \text{s}^{-1}$.

(1 Point)
Question 2  A sinusoidal perturbation

Consider a physical system with Hamiltonian $H_0$ such that
\[ H_0 |\varphi_n\rangle = E_n |\varphi_n\rangle \] (6)
with eigenvalues and eigenvectors $E_n$ and $\varphi_n$ respectively and $\langle \varphi_m | \varphi_n \rangle = \delta_{mn}$. At $t = 0$ a perturbation is applied to the system. Its Hamiltonian now becomes
\[ H(t) = H_0 + \lambda \hat{W}(t) \] (7)
where $\lambda$ is a real dimensionless parameter much smaller than 1 and $\hat{W}(t)$ is an observable of the same order of magnitude as $H_0$ and which is zero for $t < 0$.

Now assume that $\hat{W}(t)$ has the form
\[ \hat{W}(t) = \hat{W} \cos(\omega t) \] (8)
where $\omega$ is a constant angular frequency and $\hat{W}$ is a time independent observable.

a) The system is assumed to be initially in the stationary state $|\psi_i\rangle$ which is an eigenstate of $H_0$ of eigenvalue $E_i$. Calculate the state vector $|\psi(t)\rangle$ to first order in $\lambda$ and then calculate the probability $P_{if} = |\langle \varphi_f | \psi(t) \rangle|^2$ of finding the system in another eigenstate $|\varphi_f\rangle$ of $H_0$ at time $t$. (1 Point)

b) What is the transition probability induced by a constant perturbation? (i.e. $\omega = 0$) (1 Point)

c) Discuss the validity of the perturbative expansion. (1 Point)