Question 1

a) In a two-level system the transition between the ground state \( |g\rangle \) and the excited state \( |e\rangle \) has the transition frequency \( \omega_0 \).

Exactly solve the Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} |\psi_t\rangle = H |\psi_t\rangle \tag{1}
\]

with

\[
|\psi_0\rangle = \alpha(0) |g\rangle + \beta(0) |e\rangle, \quad |\alpha(t)|^2 + |\beta(t)|^2 = 1
\]

and

\[
H = -\frac{\hbar \gamma}{2} \sigma_z + \hbar \Omega (\sigma^+ + \sigma^-) \tag{2}
\]

where \( \gamma = \omega - \omega_0 \).

(1 Point)

b) Solve the Schrödinger equation using perturbation theory to first order when \( |\psi_0\rangle = |g\rangle \).

(1 Point)

Question 2

Assume that a third level \( |i\rangle \) at frequency \( \omega_1 > \omega_0 \) can be coupled to state \( |g\rangle \) via radiation. Starting from

\[
H = \hbar \omega_0 |e\rangle \langle e| + \hbar \omega_1 |i\rangle \langle i|
\]

\[
+ \hbar \Omega (|e\rangle \langle g| e^{-i\omega t} + |g\rangle \langle e| e^{i\omega t}) \tag{4}
\]

\[
+ \hbar \Omega' (|i\rangle \langle g| e^{-i\omega t} + |g\rangle \langle i| e^{i\omega t}) \tag{5}
\]

a) Find the representation in which the Hamiltonian is time independent.

(1 Point)

b) Determine the condition under which the coupling to level \( |i\rangle \) can be neglected and the system can be reduced to two levels.

(1 Point)
Question 3

Consider the dynamics of the density matrix for Question 1

\[ \frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H, \rho] + \Gamma(\sigma \rho \sigma^+ - \frac{1}{2}\sigma^+ \sigma \rho - \frac{1}{2}\sigma \rho \sigma^+) \]  

(6)

where \( \Gamma > 0 \) and

\[ H = \frac{\hbar \omega_0}{2} \sigma_z + \hbar \Omega(e^{-i\omega t} + e^{i\omega t}) \cdot \sigma \]  

(7)

a) Determine the form of the master equation when \( H \) is moved to the reference frame which is time independent. 

(1 Point)

b) Write the optical Bloch equation

(0.5 Point)

c) Solve the optical Bloch equations for

\[ \rho_0 = \left( \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array} \right) ; \ \Omega = 0 \]  

Determine \( \text{Tr}\{\rho^2\} \) as a function of time.

(0.5 Point)

d) Solve the optical Bloch equations for \( \Gamma = 0, \ \Omega > 0, \ \Delta > 0 \) and

\[ \rho_0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \]  

Determine \( \text{Tr}\{\rho^2\} \) as a function of time.

(0.5 Point)

Question 4

Show that for a two-level system \( \rho = \frac{1}{2}(1_2 + \bar{U} \cdot \bar{\sigma}) \) with

\[ \bar{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \]  

(8)

and

\[ \bar{U} = (U_x, U_y, U_z) \]  

(9)

is a real vector \( \bar{U} \in \mathbb{R}^3 \).

Show that there is positive semi-definiteness when \( |\bar{U}| \leq 1 \). (hint: evaluate the eigenvalues of the density matrix)

(1 Point)