Quantum Optics and Cold Atoms
SoSe 2014

Sheet 4 (Part 2) 26 June 2014

(Due on 1st and 14th of July 2014)

Question 3  Born-Markov master equation

The Born-Markov master equation for an optical dipole with the transition frequency \( \omega_0 \), which is coupled with the modes of the free electromagnetic field is given by:

\[
\frac{\partial}{\partial t} \rho = \frac{1}{i\hbar} [H_0, \rho] + \frac{\gamma}{2} (\langle n(\omega_0) \rangle + 1) (2\sigma\rho\sigma - \sigma\rho\sigma - \rho\sigma\rho) + \frac{\gamma}{2} \langle n(\omega_0) \rangle (2\sigma\rho\sigma - \sigma\rho\sigma - \rho\sigma\rho),
\]

where \( \rho \) is the density matrix describing the state of the dipole at the time \( t \), \( H_0 \) governs the dipole coherent dynamics and includes the frequency shift due to the coupling with the reservoir, \( \sigma = |g\rangle \langle e| \) and \( \sigma^\dagger = |e\rangle \langle g| \).

a) Determine the mean number of photons at frequency \( \omega_0 \)

\[
\langle n(\omega_0) \rangle = \frac{1}{e^{\beta\hbar\omega_0} - 1}
\]

for \( \omega_0 = 2\pi \cdot 10^{14} \) Hz (optical domain) and \( \omega_0 = 2\pi \cdot 10^9 \) Hz (microwave domain) at \( T = 300K \). Argue (i) that for optical frequencies \( \langle n(\omega_0) \rangle \ll 1 \) and (ii) when this implies that its contribution in equation (1) can be typically neglected.  

(0.5 Point)

Tip: Here are the values of the fundamental constants (CGS units) in equation (1) where \( \beta = 1/(k_B T) \): the Planck’s constant is \( \hbar = 1.06 \cdot 10^{-27} \) erg.s, and the Boltzmann constant, which in fact was also introduced by Planck, is \( k_B = 1.38 \cdot 10^{-16} \) erg.K\(^{-1}\).

b) The damping rate in equation (1) is:

\[
\gamma = 2\text{Re} \int_0^\infty d\tau \sum_\lambda |g_\lambda|^2 e^{i(\omega_0 - \omega_\lambda)\tau}
\]

with \( \sum_\lambda := \sum_{\xi,\eta} \sum_{\xi,\eta,\xi,\eta} \) the sum over the modes of the electromagnetic field. Here \( g_\lambda = -\frac{dE_\lambda}{\hbar} \), where \( d \) is the dipole moment and \( E_\lambda = \sqrt{\frac{2\pi\hbar\omega_\lambda}{V}} \xi_\lambda \) (in Gauss-units), with \( \omega_\lambda = c|\xi_\lambda| \) and \( V \) the quantization volume. Cast the sum into an integral, valid for large volumes

\[
\sum_{\xi,\eta} \rightarrow \frac{V}{8\pi^3} \int dk
\]
and use spherical coordinates to show that $\gamma$ can be cast into the form:

$$\gamma = 2 \operatorname{Re} \int_0^\infty d\tau \int_0^\infty d\omega \ \omega^3 e^{i(\omega_0 - \omega)\tau} \left[ \alpha \int d\Omega \sum_{\xi \perp \hat{n}} |d \cdot \epsilon|^2 \right],$$  \hspace{1cm} (5)

with $\alpha = 1/(4\pi^2 \hbar c^3)$ and $\Omega$ the solid angle, which determines the direction $\hat{n}$ of the photon wave vector: $d\Omega = (d\cos \theta)(d\phi)$ with $-1 \leq \cos \theta \leq 1$, $0 \leq \phi \leq 2\pi$, and $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. \hspace{2cm} (2 Point)

c) Use that $\operatorname{Re} \int_0^\infty d\tau e^{i(\omega_0 - \omega)\tau} = \pi \delta(\omega_0 - \omega)$ and show that

$$\gamma = \frac{4 |d|^2 \omega_0^3}{3 \hbar c^3} \int d\Omega P(\Omega),$$  \hspace{1cm} (6)

where

$$P(\Omega) = \frac{3}{8\pi} \int \sum_{\xi \perp \hat{n}} |d \cdot \epsilon|^2 \left| \frac{d}{|d|^2} \right|^2.$$  \hspace{1cm} (7)

(1 Point)

d) Show that $\int d\Omega P(\Omega) = 1$ using that $\sum_{\xi \perp \hat{n}} |d \cdot \epsilon|^2 = |d|^2 - |d \cdot \hat{n}|^2$ and setting $\hat{n}$ along the $\hat{z}$ axis, so that $|d \cdot \hat{n}| = |d| \cos \theta$. \hspace{2cm} (1 Point)

Note that:

$$\gamma = \frac{4 |d|^2 \omega_0^3}{3 \hbar c^3}$$  \hspace{1cm} (8)

is the Einstein’s $A$ coefficient for spontaneous emission in the CGS (Gauss) units. In the SI units

$$\gamma_{\text{SI}} = \frac{\gamma_{\text{CGS}}}{4\pi \varepsilon_0} = \frac{|d|^2 \omega_0^3}{3\pi \varepsilon_0 \hbar c^3}.$$  \hspace{1cm} (9)

**Question 4  Unraveling the master equation**

Consider the master equation for a damped Harmonic oscillator:

$$\frac{\partial}{\partial t} \rho = \frac{1}{i\hbar} [H_0, \rho] + \kappa \left( 2a \rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a \right),$$  \hspace{1cm} (10)

where $a$ and $a^\dagger$ are the photon annihilation and creation operator, $H_0$ governs the field coherent dynamics and $\kappa$ is the photon damping rate.

Write the equation in terms of an effective, non hermitian Hamiltonian and a jump operator. Write $\rho = \sum_k \rho^{(k)}$ and determine the exact form of $\rho^{(k)}$ as the $k$-th order term in the power expansion in terms of the jump operator. Discuss the result in terms of quantum trajectories. \hspace{2cm} (3 Point)