Question 1  Jaynes-Cummings Hamiltonian

Consider the interaction between the two level atom and the cavity field described by the Jaynes-Cummings Hamiltonian:

\[ H = \hbar \omega_0 \sigma^\dagger \sigma + \hbar \omega_c a^\dagger a + \hbar g (a^\dagger \sigma + \sigma^\dagger a), \]  

(1)

where the coupling constant \( g \) is real, \( \omega_0 \) is the atomic transition frequency and \( \omega_c \) is the cavity resonance frequency. The rising and lowering operators between the ground state \( |g\rangle \) and the excited state \( |e\rangle \) are defined as \( \sigma^\dagger = |e\rangle \langle g| \) and \( \sigma = |g\rangle \langle e| \) and \( a^\dagger, a \) are the photon creation and annihilation operators.

a) Solve the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} |\psi\rangle_t = H |\psi\rangle_t \]  

(2)

for the initial condition \( |\psi\rangle_0 = |g\rangle \sum_{n=0}^{\infty} c_n |n\rangle \) with \( \sum_{n=0}^{\infty} |c_n|^2 = 1 \). Discuss the spectrum of the eigenvalues and the eigenvectors of the Hamiltonian \( H \) as the functions of the detuning \( \delta = \omega_c - \omega_0 \) and of the photon number \( n \).

(2 Point)

Question 2  Master equations for a damped harmonic oscillator

Consider an optical resonator crossed by a beam of atoms as shown on the picture. The light-atom interaction time \( \tau \) is much smaller than the inverse injection rate \( 1/r \) and the inverse coupling strength \( (g \sqrt{\langle n \rangle})^{-1} \), where \( \langle n \rangle \) is a mean photon number, i.e. the following inequalities are fulfilled: \( r \tau \ll 1 \) and \( g \sqrt{\langle n \rangle} \tau \ll 1 \).

In this case the field evolution can be described by the following master equation:

\[ \partial_t \rho = \gamma_g (a \rho a^\dagger - \frac{1}{2} a^\dagger a \rho - \frac{1}{2} \rho a^\dagger a) + \gamma_e (a^\dagger \rho a - \frac{1}{2} a a^\dagger \rho - \frac{1}{2} \rho a a^\dagger), \]  

(3)

where the oscillator damping and pumping rates \( \gamma_g = r_g g^2 \tau^2 \) and \( \gamma_e = r_e g^2 \tau^2 \) are proportional to the injection rate of the atoms in the ground state \( r_g \) or in the excited state \( r_e \) respectively.

a) For \( \gamma_e > \gamma_g \) find a time scale \( t \) till which the dynamic evaluated from equation (3) is valid, given that the cavity is initially prepared in the vacuum state with density matrix \( \rho(0) = |0\rangle \langle 0| \).

(1 Point)
b) For $\gamma_g > \gamma_e$ show that in the steady state the density matrix for the cavity field is diagonal, i.e. $\rho_{nm} = 0$ for $n \neq m$. (1 Point)

c) Derive a master equation from the Jaynes-Cummings Hamiltonian (1) under the assumption that the atoms are injected with the rate $r$ in the state $|\psi\rangle_{at} = \alpha|g\rangle + \beta|e\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. (3 Point)